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Accessible Standard Algorithms for Understanding and Equity: Multidigit and Decimal Subtraction, Multiplication, and Division

Karen C. Fuson[1], Shannon Kiebler[2] & Robyn Decker[3]

[1] School of Education and Social Policy, Northwestern University, USA

[2] Professional Development Specialist, Empower Consulting

[3] Professional Development Specialist, UltramathPD Consulting; Ottawa Area Intermediate School District, Mathematics Consultant

Correspondence: Professor Emerita Karen C. Fuson, School of Education and Social Policy, Northwestern University, USA.

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Abstract

A standard algorithm uses single-digit operations and concepts of place value (Fuson & Beckmann, 2012/2013; The Common Core Writing Team, 6 March 2015). We clarify in this paper that there are several standard algorithms for each kind of multidigit computation and summarize criteria for choosing better algorithms to teach. We discuss the better standard algorithms for multidigit and decimal subtraction, multiplication, and division that we have found to be accessible to students and to parents in our years of work across many schools across the United States. By accessible we mean that students can understand, explain, and carry out accurately the written methods. We teach students to make math drawings that facilitate their understanding by relating each step in the drawing to each step in the written accessible standard algorithm.

Keywords: math teaching, computation, standard algorithms, math drawings, explaining

1. Introduction

Through the years and across the globe, many different ways to write the steps in multidigit addition, subtraction, multiplication, and division have been taught in schools. In recent decades in the United States there has been considerable controversy about which methods to teach and when they should be taught. The research of Karen Fuson has for many years sought to find a middle ground in this controversy (see the reference list and the list of publications on her website karenfusonmath.net or karenfusonmath.com). She did many studies identifying and analyzing cognitively and mathematically methods that students developed when using manipulatives or math drawings. These methods were then tried out in many different classrooms. Two methods for each operation were chosen for inclusion in a math program called *Math Expressions* that is used widely across the United States. These methods were called accessible because they were the easiest and most accurate for students. They were also called mathematically desirable because they used different aspects of place value that could be discussed and related for deeper understanding.

These accessible and mathematically-desirable methods have now been used successfully for many years. The purpose of this paper is to share these methods worldwide so that other educators can consider using them. A related paper discusses what a standard algorithm is and then focuses on accessible standard algorithms for multidigit addition (Fuson, Kiebler, & Decker, 2024). That paper is available on the NCTM website nctm.org. Teaching materials discussed in that paper and a lesson plan for using these materials to compare multidigit numbers is also on nctm.org.

A standard algorithm uses single-digit operations and concepts of place value (Fuson & Beckmann, 2012/2013; The Common Core Writing Team, 6 March 2015). Because standard algorithms have often been taught rotely with little understanding, some educators in the United States have opposed teaching standard algorithms at all or insist on teaching them after students have had years of exploring and using simpler strategies. They assert that this approach leads to better understanding by students, but there is no evidence for this position. The Common Core State Standards (National Governors Association Center for Best Practices (NGA Center) and Council of Chief State School Officers (CCSO), 2010) state that students should develop, discuss, and use efficient, accurate, and generalizable methods when they first encounter each multidigit operation. But these methods are not to be learned just as symbolic copying of a teacher's steps. Students are to use concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. They are to relate the strategy to a written method and explain the reasoning used. The accessible and mathematically-desirable methods discussed in this paper are developed in these meaning-making ways.

2. Criteria for Choosing Algorithms in Classrooms

Fuson and Beckmann (2012/2013), Fuson and Li (2014), Fuson (2020), and Fuson, Kiebler, and Decker (2024) identified criteria for choosing which algorithms should be emphasized in the classroom for multidigit addition and/or subtraction. We expanded these criteria to all four operations in Table 1 below. Important conceptual or procedural goals for algorithms are identified in the first column with the criterion that follows from that goal in the second column. Choosing algorithms that meet many of the conceptual or procedural goals and do not stimulate students to make errors is important. In our many years of work in classrooms we found that some students preferred different accessible algorithms, so we introduced in classrooms our two best such accessible algorithms for each operation. Having two such methods enriched the mathematical discussion as the methods were contrasted, and those discussions helped to emphasize that reasoning and explaining is an important part of mathematics. The criteria in Table 1 that each accessible algorithm meets will be discussed in the section on those algorithms.

Table 1. Conceptual or Procedural Goals with Their Criteria for Standard Algorithms That Are Accessible and Should Be Taught

Conceptual or Procedural Goals	Criteria for the Algorithm
1. Understanding and using place value is crucial so	The algorithm supports and uses place value correctly by
a. Adding/subtracting like units is visually easy	Aligning like places (like units)
b. Students can easily add the teen total to the correct place values	Making it easy to see in the correct places any teen totals for the new grouped unit
c. Errors are not introduced by visual confusion	Making it easy to see where to write the new grouped unit
d. Adding results of steps is important	Writing the largest partial result first so other partial results can be written below it and aligned more easily
e. Correct place values are used for all steps	Writing place values of numbers in the problem or partial results in expanded notation
2. Eliminate errors created by an algorithm writing numbers not in their usual order (e.g., for 14 write the 4 and then the 1)	By allowing students to write teen numbers in their usual order (e.g., for 14 write the 1 and then the 4)
3. Students do single-digit computations accurately	By making single-digit computations easy
4. Prevent errors introduced by alternating different kinds of steps because students continue one kind of step instead of changing to the alternate step	By doing all of one kind of step first and then doing the other kind of step and not alternating steps
5. For conceptual clarity keep the result separate from the original problem	By leaving the initial multidigit numbers unchanged
6. Allow students to calculate in preferred approaches such as from left to right in the same direction as they read words	By being able to be solved by going left to right

Examples of math drawings, accessible algorithms, and student explanations are given in Figure 1 for multidigit addition and Figure 6 for multdigit subtraction. These methods exemplify several of the goals and criteria in Table 1. In Figure 1 criterion1a is shown by step a where the numbers align place values by putting one number above the other. Criteria 1b, 1c, and 2 are shown in the Figure 1 steps b and c. Teen numbers can be seen as teen numbers because they are written near each other and aligned below the numbers in the problem, and they can be written in their usual order because the teen number is so close to the ones number. Criterion 3 is shown in steps c and d where it is easy to add the larger numbers first and then increase that total by 1. Criterion 5 is shown in the Figure 1 steps b, c, and d where no numbers are written above the problem or other than below where the total is accumulating. In Figure 6 criterion 6 is shown by doing the steps from left to right.

We emphasize in classrooms that students need to use correct place-value language when they are discussing and explaining multidigit methods because we want the written symbols to take on place-value meanings. This may be especially important for students speaking European languages where place-value language is not as clear as it is in East Asian languages (e.g., in English *fourteen* instead of *ten four* for 14 and *forty* instead of *four tens* for 40). We find that it is important for English-speaking children to use place-value words like *hundreds*, *tens*, and *ones* when discussing multidigit operations (see the examples in Figures 1 and 6). However, using correct place-value language is important in all countries because the existence of mathematical or place-value concepts in the language does not mean that those concepts are in the heads of students: they may not yet understand some meanings. For example, Ho and Fuson (1998) found that Chinese-speaking kindergartners were better than were English-speaking kindergarten children at understanding that the numbers 12, 15, 17, and 19 referred to ten things and 2, 5, 7, and 9 things. But a substantial proportion of the Chinese-speaking children did not understand the ten things in these numbers even though the words for these numbers contained the word for ten (e.g., *ten two*). Because Chinese children who understood the embedded ten could count to a much higher number, learning the counting word sequence at least to 70 seems to help children understand the embedded-ten cardinality involved in the teen counting words. Another example occurred for Spanish when we were interviewing children about the meaning of the 1 in 16. The child first pulled out one object but then pulled out ten objects while saying in a wondering tone, “Oh, dieciseis es diez y seis.” [The counting word *dieciseis* is the cardinal numbers *ten and six*.”)

There are also language issues about what to use for the steps in adding and subtracting where place-value units are grouped or ungrouped. Physical words like *bundled, traded,* or *borrowed* are sometimes used. We found that the word *grouped* for addition and *ungrouped* for subtraction were explicit and general enough to relate to the steps with the math drawings and to the steps students were doing with the mathematical symbols.

3. Multidigit Addition

As background for the accessible standard algorithms in this paper, we start here with a figure that shows the math drawings, written method, and explanation of a student doing the most important accessible standard algorithm for multidigit addition. We do this to emphasize that our approach to algorithms is always focused on meanings. Students make math drawings of place-value quantities that relate to their written method. Students explain their methods and then classmates ask questions, leading to productive discussions.

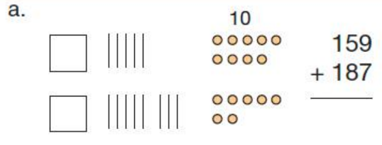
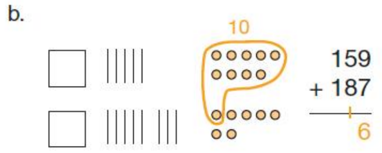
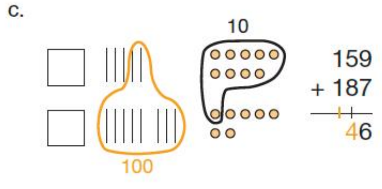
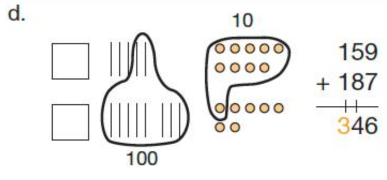
Math Drawing and Problem	Explanation Using Place-Value Language About Hundreds, Tens, and Ones
a. 	I drew one hundred, five tens, and nine ones to show one hundred fifty nine, and here below it I drew one hundred, eight tens, and seven ones for one hundred eighty seven. I put the ones below the ones, the tens below the tens, and the hundreds below the hundreds so I could add them easily.
b. 	See here in my drawing, nine ones need one more one from the seven to make ten ones that I circled here, and I wrote 10. That leaves six ones here. With the numbers the seven gives one to the nine to make ten that I write over here in the tens column, see one ten. And I write six ones here in the ones column.
c. 	With the tens, I start with eight because it is more than five so it is easier. I get two tens from five tens to make ten tens, see here, and I write one hundred here to remind me that the ten tens make one hundred. There are three tens left in the five tens and I have one more ten from my ones (see here in my drawing and the one ten at the bottom of the tens column). That makes four tens and the one hundred. So in my problem I write the one hundred below in the hundreds column and the four tens in the tens column.
d. 	There are three hundreds, two in the original numbers I'm adding and one new hundred from the ten tens. I write three hundreds here in the hundreds column. Are there any questions? Yes, Stephanie.
Student Question	**Explainer Answer**
Stephanie: For the tens, you never said fourteen tens as the total of the tens. Why not?	Because when I'm making ten tens, I just can write that one hundred over here with the hundreds and just think about how many tens I need to write. But I can think eight tens and five tens is thirteen tens and one more ten is fourteen tens, so that is one hundred and four tens. You can do it either way. (Aki)
Aki: Do you still need to make the drawings or did you just make them so you could explain better?	I don't have to make the drawings, but I can explain better with a drawing because you can see the hundreds, tens, and ones so well. (Jorge)
Jorge: Do you do make-a-ten in your head or just know those answers?	I just know all of the nine totals because of the pattern: the ones number in the teen number is one less that the number added to nine because it has to give one to nine to make ten. So nine plus seven is sixteen. I just know that pattern super fast. For eight plus five, I do make-a-ten fast, sort of just thinking five minus two is three, so thirteen. (Sam)
Sam: I know five and eight is thirteen, so why did you write a four in the tens column, Karen?	Because I had one more ten from the ones. See here in the drawing: nine ones and one one from the seven ones make ten ones. I wrote 10 here to remind me, and here in the problem I wrote the new one ten below where I can add it in after I find thirteen. You have to write your new one ten big enough to be sure you see it.
Sam: Oh yes, I see it now. I can see the new one ten when I write it, but I couldn't see yours.	OK, thanks. I'll write it bigger next time so everyone can see it.

Figure 1. Math Drawings and Explanation of an Accessible Standard Algorithm for Multidigit Addition

We also show in Figure 2 the Secret Code Cards that are important conceptual supports to help students to understand place-value notation in algorithms. Pages of the Secret Code Cards and directions to make them are on karenfusonmath.net or karenfusonmath.com in the file *How to make Secret Code Cards to show place-value* *numerals* on the Math Expressions Users page. A page of ten-structured dots and directions for helping students to make math drawings for multidigit addition and subtraction are on karenfusonmath.net or karenfusonmath.com in the file *How to make math drawings for hundreds, tens, and ones* on the Math Expressions Users page. This page of the website also contains directions to use both of these learning supports to understand place-value notation and to compare 2-digit and 3-digit numbers (*see Comparing multidigit numbers using math drawings and Secret Code Cards*) and to add and subtract multidigit numbers (see *Adding and subtracting numbers using math drawings and Secret Code Cards*).

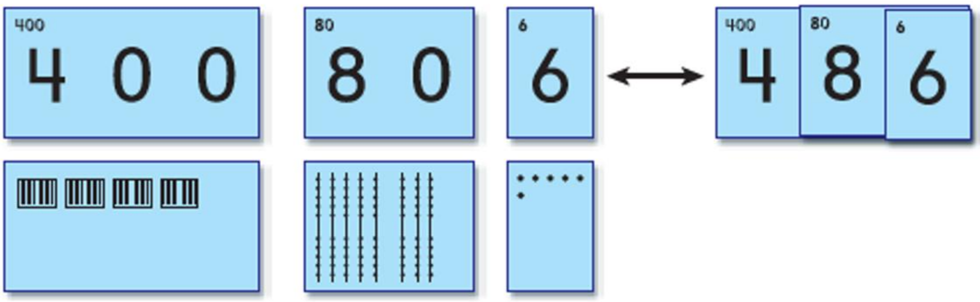


Figure 2. Secret-Code Cards Layer and Unlayer to Connect Single Digits to Place-Value Expanded Notation

Figure 3 shows our two accessible standard algorithms and a standard algorithm that is erroneously taken by many people in the United States as THE standard algorithm that must be taught. The New Groups Below accessible standard algorithm was explained and related to math drawings in Figure 1. New Groups Below meets six desirable criteria from Table 1 (1a, 1b, 1c, 2, 3, 5) and New Groups Above meets only one criterion (1a). New Groups Above creates difficulties for students when adding the single digits in a given place because they either must add the 1 above to the top number and add a number they cannot see (9) while ignoring a number they do see (8) or add the two larger numbers first (e.g., add 8 and 5) and then remember to add the 1, which many forget because it is up above the 8 and the 5. The Show All Totals accessible standard algorithm meets all 10 criteria in Table 1. It can be done right to left instead of left to right as we show it, in which case it does not meet criteria 1d and 6. But Show All Totals does get cumbersome for large numbers so it can be replaced by New Groups Below if desired. In discussions the Show All Totals algorithm helps students see the place values of the larger numbers, and the New Groups Below algorithm helps students see how the same steps are repeated for each place.

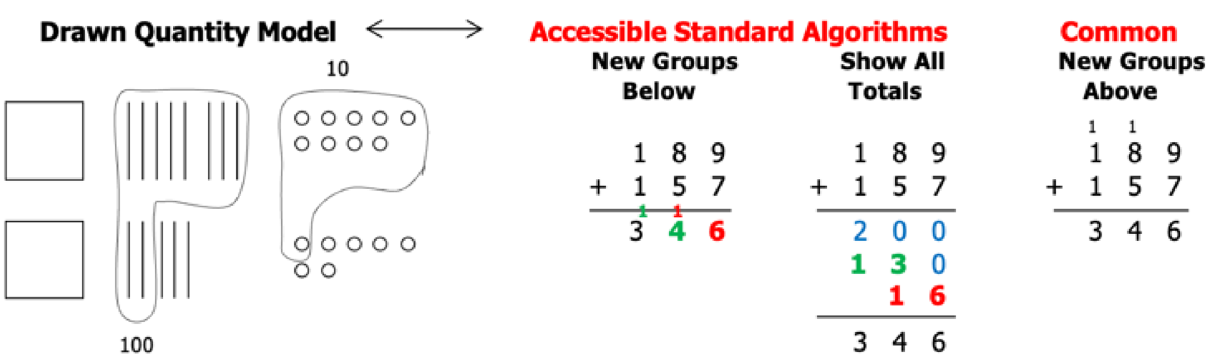


Figure 3. Multidigit Addition Math Drawings, Two Accessible Standard Algorithms, and a Common Standard Algorithm That Is Flawed

4. More Information About the Accessible Algorithms for All Operations

More detailed descriptions of the addition accessible standard algorithms can be found in Publications on the website karenfusonmath.net or karenfusonmath.com (Fuson, 2020, Fuson & Beckman, 2012/2013, and Fuson & Li, 2014). More detailed discussions of the accessible standard algorithms for multidigit multiplication and division can be found in Fuson (2003a) and Fuson (2003b). The accessible standard algorithms discussed in both papers meet the U.S. Common Core State Standards (NGA Center and CCSO, 2010) and other high-quality state standards.

Additional resources for understanding the accessible standard algorithms for all four operations can be found in Table 2. All of the students shown in the videos are from Spanish-speaking backgrounds and many are from backgrounds of poverty. They are wearing uniforms because their public school requires them to do so to decrease gang activity.

Table 2. Detailed Explanations of Accessible Standard Algorithms

What to see	Where to go
To watch first graders explaining math drawings and the three multidigit addition standard algorithms in Figure 3	Please go to https://karenfusonmath.net/classroom-videos/#C-Longer-Classroom-Teaching-Examples and play Grade 1
To watch the first author explaining math drawings and the three multidigit addition standard algorithms along with students from several grades solving problems	Please go to https://karenfusonmath.net/classroom-videos/#B-Math-Explanations and play Multidigit Addition
To watch third graders explain the 3-digit subtraction accessible standard algorithms	Please go to https://karenfusonmath.net/classroom-videos/#C-Longer-Classroom-Teaching-Examples and play G3 Multidigit Subtraction.
To watch fifth graders explain 7-digit subtraction accessible standard algorithms	Please go to https://karenfusonmath.net/classroom-videos/#G-Place-Value-and-Multidigit-Addition-and-Subtraction and play the last video G4 Explaining 7-digit subtraction.
To watch fourth graders explaining all three multiplication accessible standard algorithms	Please go to https://karenfusonmath.net/classroom-videos/#C-Longer-Classroom-Teaching-Examples and play the fourth video 4 G4 Multidigit Multiplication.
To watch the first author explaining math drawings and the three multiplication accessible standard algorithms with examples of students using each algorithm	Please go to https://karenfusonmath.net/classroom-videos/#B-Math-Explanations and play Multidigit Multiplication
For a description of extensions of the multidigit accessible standard algorithms to decimals	Please go to https://karenfusonmath.net/teaching-progressions/ and scroll down to Numbers Base Ten, Place Value Parts 4, 5, 6, 7 and read Beckmann and Fuson, April, 2014, in Presentations

5. Multidigit Subtraction

The accessible standard algorithms for multidigit subtraction are shown in the middle of Figure 4 below. A math drawing that can direct all of the methods is shown on the left. A standard algorithm that is erroneously taken by many people in the United States as THE standard algorithm that must be taught is shown on the right. The two accessible standard algorithms are shown in the middle. They have the same approach but operate in different directions.

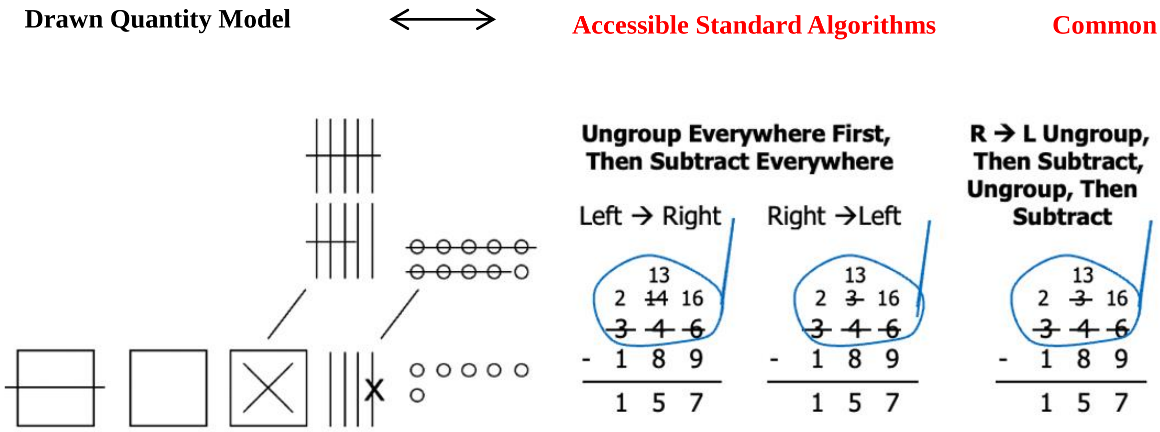


Figure 4. Multidigit Subtraction Math Drawings, Two Accessible Standard Algorithms, and a Common Standard Algorithm That Is Flawed

The common standard algorithm shown on the right alternates the two main operations: students ungroup if needed, then subtract, then ungroup if needed, then subtract, etc. This alternating leads students to make errors. As shown in Figure 5, after ungrouping and subtracting in the ones column, the student moves to the next left column. The student is in subtract mode, sees 5 and 3, subtracts to get 2, and writes 2, which is wrong. The answer should be 8 from 13 – 5 = 8. The student knows how to ungroup and has just done so in the ones column. So the alternating steps require that the student must be super aware of checking to see if ungrouping to get more in a given column is needed and do so for each new column.

The error of subtracting a smaller top number from a larger bottom number is very frequent and can be difficult to overcome. That is why we devised the initial step shown in Figure 4 of drawing a big loop around the top number and having students write the ungrouping within that loop. They draw a little stick to that loop and call the whole thing a *magnifying glass* that reminds them to check whether every column has a top number greater than or as great as the bottom number so that they can subtract. This loop also serves as a conceptual reminder that is discussed by students: The ungrouping does not change the value of the original top multidigit number (the total) but just rewrites it in an equivalent form. The loop also serves to highlight the total as the starting point for subtraction and so the math drawing only shows this total. Subtraction can be done by drawing the total and the known addend and then matching to find how many more the total has than the known addend. But this approach gets messy, and we found that students using it confused subtraction with addition.

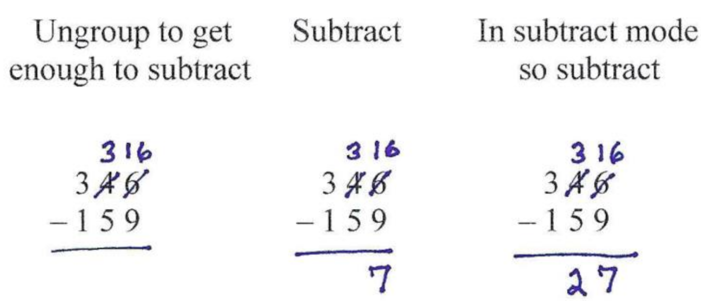
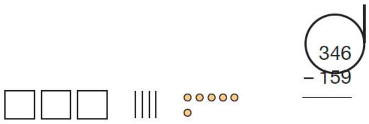
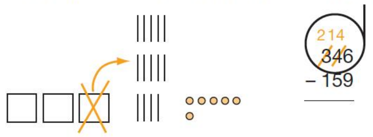
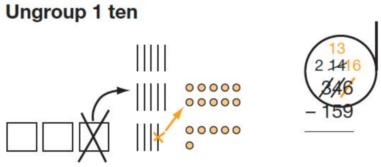
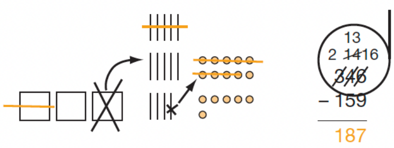


Figure 5. The Error Created by the Common Standard Algorithm Because of Its Alternating Steps

The subtraction accessible standard algorithms are variations of each other in which a student ungroups from the left or from the right. Students do any needed ungrouping first and then do all of the subtracting. Ungrouping from the left may be unfamiliar, so Figure 6 shows the steps in such ungrouping, along with the math drawings that support it and the explanation by the student using this accessible standard algorithm. For both the left-to-right and the right-to-left Ungroup Everywhere First as Needed, Then Subtract Everywhere accessible standard algorithms, students can subtract from the left or from the right after they have ungrouped. Some students like to vary how they solve.

Math Drawing and Problem	Explanation Using Place-Value Language About Hundreds, Tens, and Ones
a. **Write problem and make drawing** 	I drew three hundreds, four tens, and six ones to show three hundred forty six. I wrote one hundred,five tens, nine ones below three hundred forty six in my problem, but I did not draw it because it is already part of three hundred forty six. To subtract, we separate the total into two numbers, the number we are taking away and the number that is left. Here I drew my magnifying glass around the total to remind me to check if I need to ungroup to get more to subtract.
b. **Ungroup 1 hundred (solving left to right)** 	I checked to see if I need to ungroup left to right. I can do it right to left too. So here in the hundreds I can take one hundred from three hundreds, so that column is OK. In the tens column, I cannot take five tens from four tens because five is more than four. So I need to get more tens to go with my four tens. I open up one hundred to make it be ten tens. Here I wrote my ten tens in two rows of five so you can see them clearly. And in my problem I showed that ungrouping by crossing out the three hundreds and writing the two hundreds I have left. And the four tens become fourteen tens here. So I'll be able to subtract the tens.
c. **Ungroup 1 ten** 	Now I check to see if I can subtract the ones. Nope. Nine is more than six, so I need to get more ones also. I open up one ten here to show that it has ten ones hiding in it. I write them in two rows of five so I know I made exactly ten and you can see them. In my problem I ungrouped by taking one ten from the fourteen tens and writing thirteen above in the tens column. And the ten ones make sixteen ones with the six, so I write sixteen at the top of the ones column.
d. **Subtract H, T, O L to Rt or Rt to L** 	Now I can subtract in every column. I can go in either direction. I'll go left to right again. I take away one hundred in my drawing, and one hundred is left. My problem agrees: I take away one hundred from the two hundreds and write the one hundred that is left. I'll subtract five tens from thirteen tens and get eight tens. I just know that. But here in my drawing I'll take the five tens from the ten tens, and I can do make-a-ten if I don't know thirteen minus five. See, five more left in the ten and the three in thirteen make eight. For the ones I can use Karen's pattern she just explained, that the teen total is one less than the ones added on to a nine. So sixteen minus nine is seven. See here in the drawing, you can see the one extra with the nine that gets added to the six ones to make seven ones. Are there any questions? Yes, Sybilla?

Student Question	Explainer Answer
Sybilla: Doug, why didn't you subtract six ones from nine ones to get three ones in the answer?	Because we have to subtract the addend from the total. Six is part of the total, so we have to subtract from it. But we can't, so that's why I had to get more ones here. Good question, even though I know you know this. Hank?
Hank: What if you checked your hundreds and the bottom number was bigger? How could you subtract?	I couldn't. The total has to be bigger than the addend I subtract because that addend is just part of the total. But sometimes I write the numbers backwards, so I check the problem again if I can't subtract the hundreds. Efrain?
Efrain: How would your problem be different if you had ungrouped right to left?	Only the tens place would look different. Remember how we did it both ways and talked about this yesterday? And look at Yeping's problem. He ungrouped right to left. The tens place looks different because you ungroup one ten to make ten ones before you get ten tens. So you write three and then thirteen. But I end up with thirteen, so the ungrouping gets the same number in each column ready to subtract.

Figure 6. Math Drawings and Student Explanation of the Accessible Multidigit Subtraction Standard Algorithm Ungroup Everywhere First as Needed, Then Subtract Everywhere

The main advantage of the two accessible standard algorithms is that they do not alternate the two main aspects of multidigit subtracting: ungrouping where needed to get enough to subtract and actually subtracting in a given place value. One also is done from the left, which many students prefer. The common standard algorithm has neither of these positive attributes, and it stimulates students to make errors by alternating the ungrouping and subtracting major steps.

One common subtraction algorithm that comes from some countries in Latin American and Europe is particularly tricky (see Figure 7) and can become confused with the common subtraction algorithm (Ron, 1998). We have been in classrooms where it took parts of two classes to understand and explain this tricky algorithm that may come from homes of students. In our experiences neither parents nor teachers understood why steps were done; they had just learned how to do those steps. See Ron (1998) in the Publications page on karenfusonmath.net or karenfusonmath.com for more discussion of how this method works and how students do a mixed erroneous method that uses parts of this method and of the common standard algorithm.

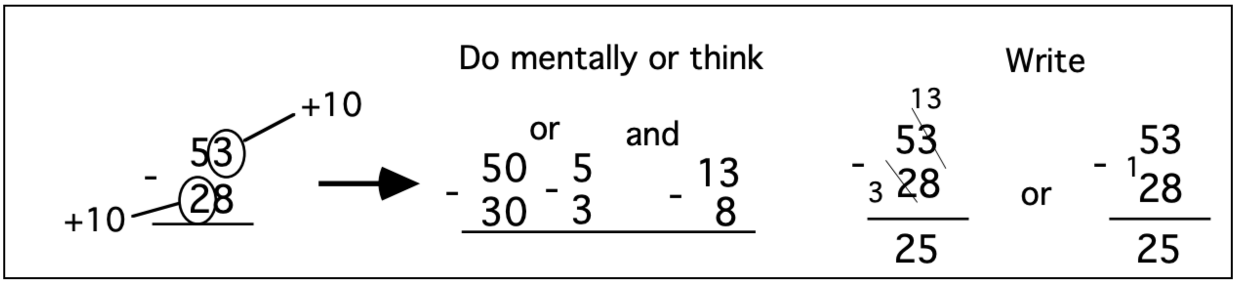


Figure 7. The European/Latino algorithm add ten to both numbers

6. Multidigit Multiplication

In our approach, array and area models are used in Grade 3 to show multiplication of single-digit numbers. In Grade 4 these models are extended to show multidigit multiplication and how place values of ones, tens, and hundreds work in multiplication. An outline of major aspects of this approach are shown below. Students can work on dot paper with the dots very close together; we use lengths between dots of 4 mm. You do not need much of the dot paper because students will soon be making sketch models of multiplication problems on plain paper and relating these models to written methods.

First, students draw an area to show multiplication of ones units. They draw the area model and then draw in the squares to show the area. Students can draw arrays or area. We suggest using area throughout because area is more difficult for students and this will give them a lot of experience with imagining the squares that make the area. It is helpful for students to write the lengths on all four sides of the rectangle to see all of the relationships involved with larger numbers.

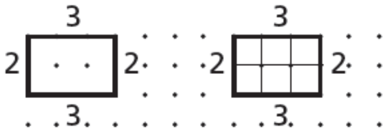
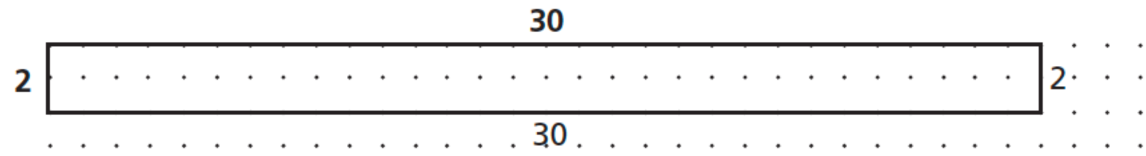
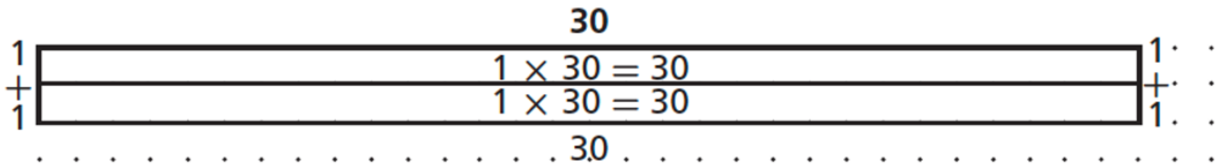


Figure 8. Area drawings for single-digit multiplications

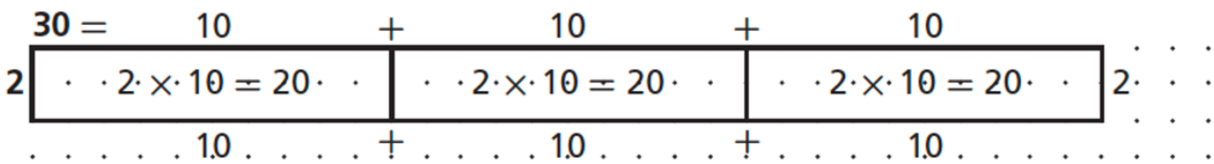
Next students draw an area model for ones times tens. They draw multiple copies of that area model and then explore and find relationships within that area model. Below are shown three patterns that are important for students to see and discuss. The third model is the crucial model because it shows the pattern *ones units times tens units makes a product of tens units*. The unit being counted here is a 1 x 10 rectangle.



- Divide the rectangle *across* to show 2 groups of 30.



- Divide the rectangle *up-and-down* to show 3 groups of 20.



- Divide the rectangle both *across* and *up-and-down* to show 6 groups of 10. (Students need only label one of the inner rectangles.)

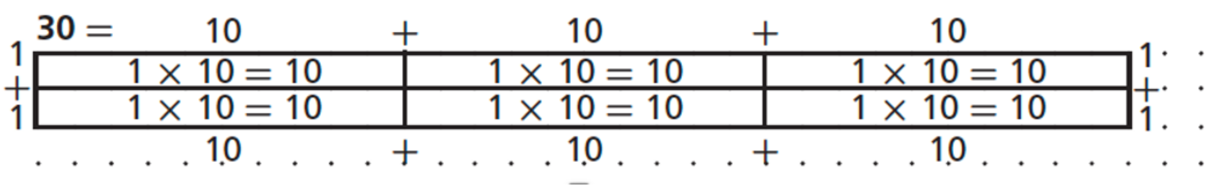


Figure 9. Patterns in area drawings for ones times tens multiplications

Next students need to explore the *tens units times tens units* case to see that they can make units of 100 and that they get as many hundreds units as are made by the products of the digits in the tens places. In all of this work it is important that teachers support students to be explicit about which units they are using. The math drawings are very helpful in understanding and discussing units.

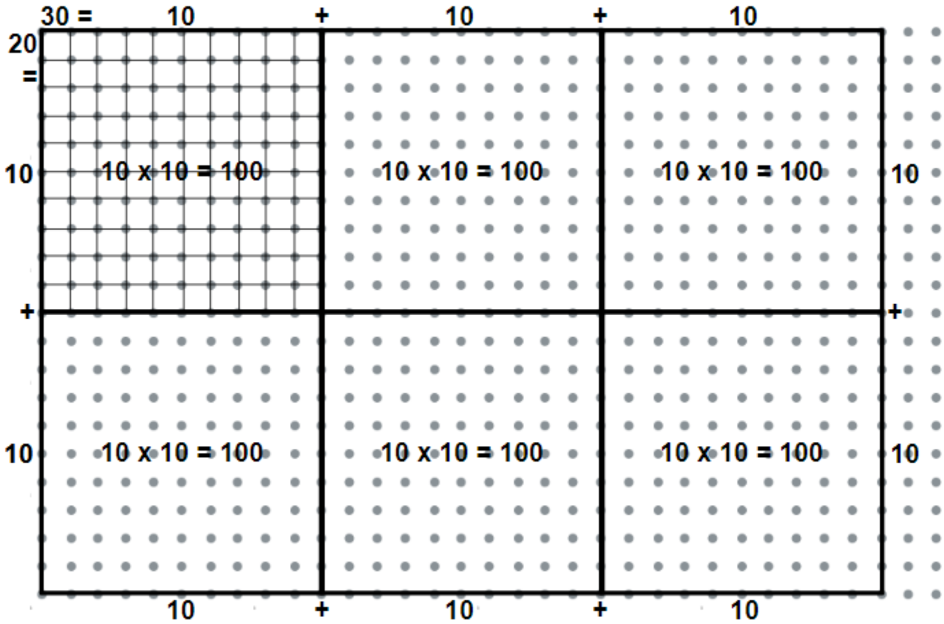


Figure 10. Units and patterns in area drawings for tens times tens multiplications

The next step is for students to discuss the patterns they see in multiplying *ones units times ones units*, *ones units times tens units,* and *tens units times tens units*. These patterns can be combined into a table and discussed. The patterns are easy because the product has as many zeroes as are in the factors. This is because each zero in the multiplying factor moves the multiplied factor one place to the left as it is multiplied by ten for each place that has a zero. Students do need to discuss the special case of multiplying by five because it seems to violate the patterns they have just found. But this is because five times any even number ends in zero, so that adds an extra zero to the product of the places. Students can discuss how they can move from column B to column C in Table 3 by using the associative and commutative properties.

Table 3. Patterns in multiplications involving zeroes

A	B	C	D
2 x 3	2 x 1 x 3 x 1	6 x 1	6
2 x 30	2 x 1 x 3 x 10	6 x 10	60
20 x 30	2 x 10 x 3 x 10	6 x 100	600

Most students rather quickly can move from drawing on the dot paper to making sketches on plain paper or dry erase boards in which they show the tens and the ones lengths in the factors and find the products inside each part of the area model (See Figure 11). Students can then write the totals out at the side and add them to find the total area as shown below on the right of the drawing. Because the goal of drawing a model is to stimulate a written method that makes sense as it is related to the model, students need to try writing a written method and explain and justify it by relating it to the model. Student explaining and justifying needs to use correct place-value language. For example, for 20 x 30 students need to say 2 *tens* times 3 *tens* is 6 *hundreds* or *twenty times thirty is six hundred* and not just say 2 times 3 is 6.

Shown in the second row of Figure 11 is the Expanded Notation method developed by students so that they understand fully what they are doing. The expanded notation of 28 at the top right and the factors listed on the bottom left in blue are included initially because some students need to see these to do the correct multiplications. These steps can be dropped when students no longer need those steps, resulting in the Partial Products method.

These methods are all standard algorithms because they meet the definition of using multiplication of single digits and meanings of place value.

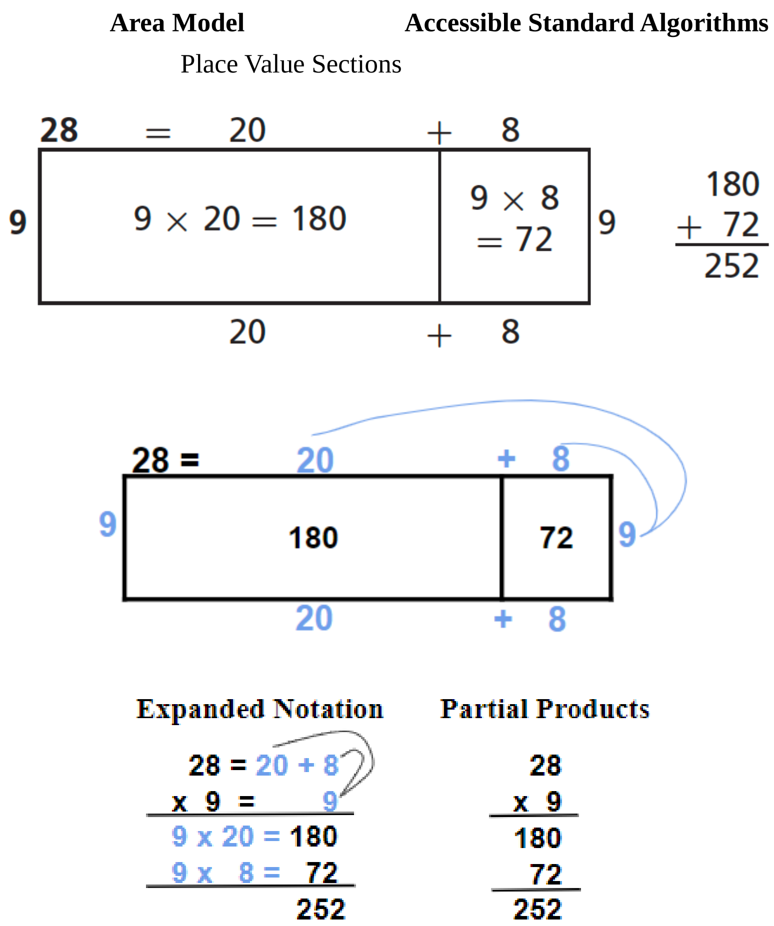


Figure 11. Area drawings and accessible standard algorithms for ones times tens multiplications

The area model that shows multiplication of 2-digit times 2-digit numbers has two rows as shown in Figure 12. Initially students find it helpful to write the factors as well as the products inside the sections of the area model. But as shown in the second area model, many students soon can write the product without needing to write the factors inside the place value sections. Some students then find the product by writing the products of the place value sections out to the right and adding them. Some students do this below the written problem as shown in Figure 12. Many students can go on to the Expanded Notation and other methods shown below. But some students find the layout in all of the written methods below so difficult that they can only be accurate if they draw a quick area model, write partial products inside, and then write the totals out to the side and add them as in the Place Value Sections method.

The Expanded Notation method in Figure 12 was developed by students so that they understand fully what they are doing. The expanded notations of 43 and 67 at the top right and the factors listed on the bottom left in blue are included initially because some students need to see these to do the correct multiplications. These steps can be dropped when students no longer need those steps, resulting in the Partial Products method. These methods are all standard algorithms because they meet the definition of using multiplication of single digits and meanings of place value.

More difficult methods are showed in the bottom row of Figure 12. In these methods the top number in the area model and in the written methods is kept as a 2-digit number and only the multiplying number, here 67, is separated into its tens and ones. This allows the area model and the written methods to have just two rows. We call all of these standard algorithms *difficult* because they involve collapsed place-value sections. We use the

New Groups Below addition method for adding because it is the easiest addition accessible standard algorithm. The non-alternating New Groups Below multiplication method shown first is easier than the other methods because one does all multiplications first and then all additions. Also, in the written method you can see each product 7×3 = 21 and 7×4 = 28 and 6×3 = 18 and 6×4 = 24. In the other two difficult alternating methods one multiplies one place then multiplies the next place and adds in any part of the first partial product that was more than one digit. These methods are more confusing because you cannot see all of the partial products and what has been added is not so clear (such confusing digits are in red). In the final method the 1 at the very top is written in the tens place but it is actually a 1 hundred coming from the 60×3 = 180. We see no reason to introduce the difficult alternating methods to students. The expanded notation or partial products methods seem clear and sufficient for students. And for those students who need to draw the area model sketch to find the partial products, it seems much better to let them do so because it supports understanding as well as correct answers.

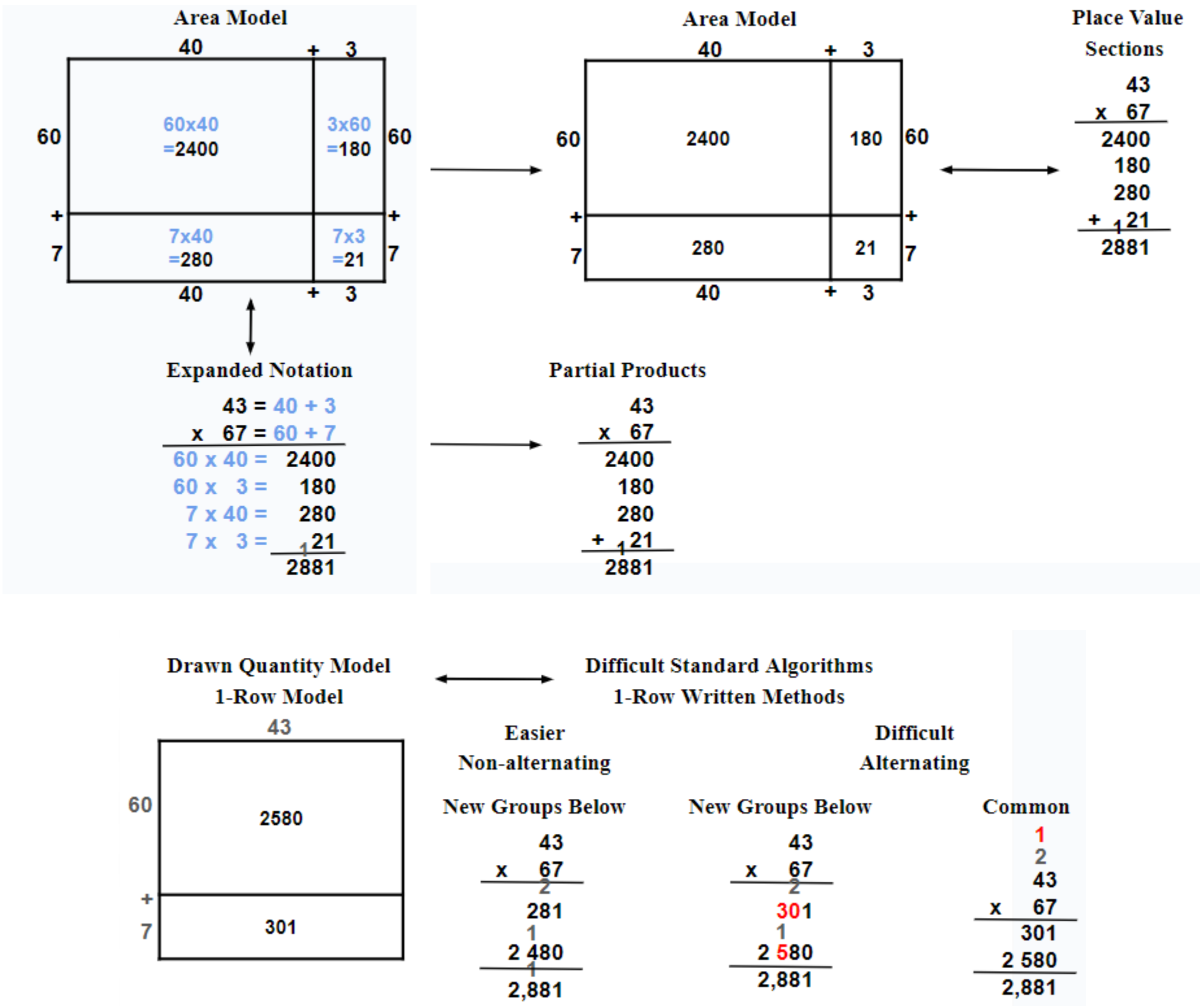


Figure 12. Area drawings and accessible standard algorithms for tens units times tens units multiplications

The two accessible standard algorithms for multidigit multiplication are shown in Figure 13. These were both discussed above. The Place Value Sections accessible algorithm uses the area model to organize the multidigit multiplications for students who find it difficult to do so in the Expanded Notation layout. With time and practice, the Expanded Notation algorithm can reduce to the Partial Products algorithm shown in Figure 12. The 1-row algorithm is one of the difficult alternating methods described above. Many people in the United States erroneously think it is THE standard algorithm and must be taught. It is difficult, and the tens units times ones units product (here 60 x 3) is written in the wrong place (see the 1 from 180 at the top is in the tens place instead of in the hundreds place where it belongs). We see no reason for this difficult method to be taught at all unless it comes from home and then it will need to be discussed and explained. Because we found students who had trouble even with the Expanded Notation layout, teaching the even more difficult 1-Row method is not at all sensible.

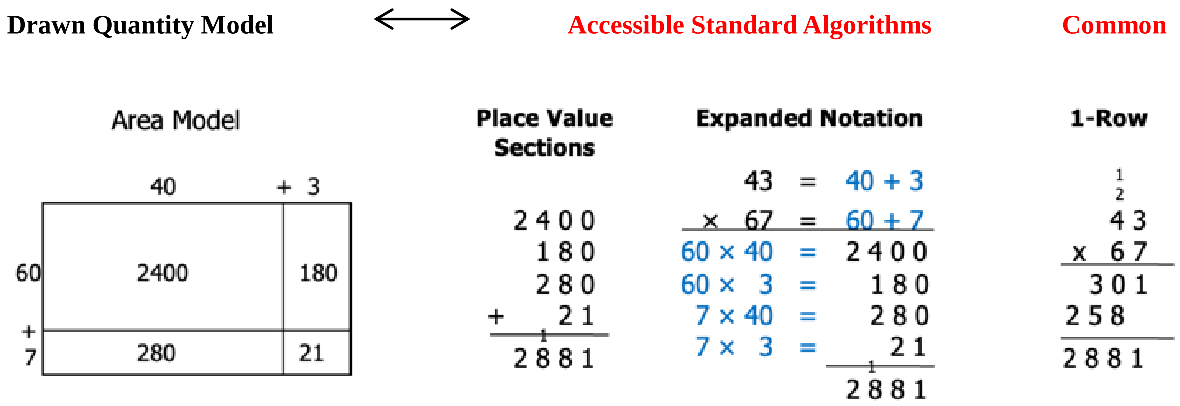


Figure 13. Multidigit Multiplication Math Drawings, Two Accessible Standard Algorithms, and a Common Standard Algorithm That Is Flawed

7. Multidigit Division

It is important for understanding division that it be related to multiplication. This can be done by using the multiplication area drawing for division but adapting the area model to find the unknown factor that is the length of the rectangle (see Rectangle Sections in Figure 14) rather than finding the area as in multiplication. The two rows in multiplication are collapsed into one row to show the known factor as the divisor, here 67. The multiplying factors for each place value that will make the unknown factor are written above the area model and are added at the end to find the unknown factor, the answer for division. Partial products are written inside each rectangle section in turn, then subtracted from the amount left at that time. The difference is written below the area model and then inside the next section of the area model to show how much area is still available for the next unknown factor length. Rectangle Sections is both a drawn quantity model and an accessible standard algorithm. It is easier for some students because it organizes the partial products they are subtracting in a sensible way. The other standard algorithms for division shown in Figure 14 use the common long division format in which the rectangle sections are below each other. These formats lose the sense of division as using the area of a rectangle although the area model still can help with making sense of the steps in these algorithms by showing each chunk as a section of the area model.

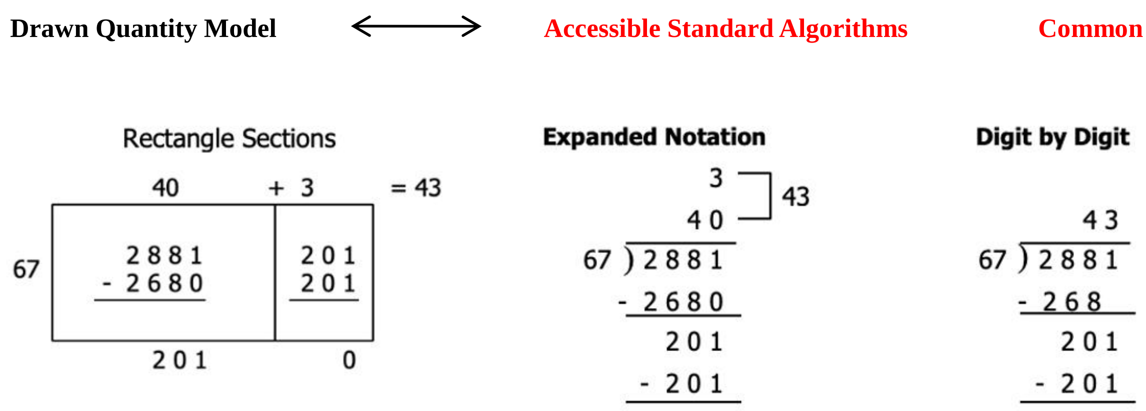


Figure 14. Multidigit Division Math Drawings, Two Accessible Standard Algorithms, and a Common Standard Algorithm That Is Flawed

The common digit-by-digit standard algorithm does not show the place values involved in the steps, so it is too often developed without understanding. It is so easy to write the places using zeroes within the long division format as in the Expanded Notation method that it seems better for students to do that, at least initially, so that they can think about and discuss the place values involved.

The two accessible standard algorithms for division support and use place value correctly by using expanded notation for the unknown factor in the tens place (here, 40). This supports students to find the correct partial product when they multiply the known factor. It also helps students to align the partial product correctly under the known product (the dividend). Writing the first part of the unknown factor (here, 40) in the vertical division format is difficult for many students. Writing the whole expanded notation for that number as 40 instead of 4 helps with the choice of where to write that number because the 0 is in the ones place. This choice is more difficult for the Digit by Digit common standard algorithm, and students must be careful to write digits in the correct aligned places. The single digits in the Digit by Digit algorithm also make it more difficult for students to understand that they are taking away parts of the original product until they cannot take any more parts away.

All three standard algorithms are done from the left, finding the largest partial product first and subtracting it. All three standard algorithms also alternate multiplying and subtracting so that the amount left for the next partial product is clear. For the second and third methods, we found that some students could not deal with the spatial-visual demands of simultaneously multiplying and aligning that partial product below the total so far. We suggested that such students write the multiplication out at the side so that they could then focus on the place values when they put that partial product back within their chosen long division format.

In working with students in many classrooms we found that multidigit multiplication and division were similar in that the common format for recording the steps in multiplication or division were too complex for some students to follow. Such students did understand the area model and could enter partial products into the area model and then find the unknown product or the unknown factor. So, the accessible standard algorithms Place Value Sections and Rectangle Sections introduced meaningful formats that gave some students success that was not possible with the common written formats.

8. Generalizing Algorithms to How Many Places?

We have seen in many classrooms that extending the addition and subtraction methods to many places is easy. Such an extension shows the generalizability of place value to adjacent places and makes students feel powerful. But extending multiplication to more than 1-digit times 4-digit numbers or 2-digit times 2-digit numbers introduces many more sources of error and adds little to the understanding of the processes involved. Likewise extending division beyond 2-digit divisors or 4-digit dividends seems pointless and even problematic. Therefore, it seems that little will be gained by having students spend time on larger problems, especially when there are many other topics that are worthy of time and attention.

9. Introducing Accessible Standard Algorithms at Any Grade

Teachers have told us that students from middle school and even high school have benefitted from discussing and using the accessible standard algorithms instead of the more difficult common standard algorithms. Some teachers approach the addition and subtraction methods by saying that students are going to use their place-value understanding to explain how different computation methods work. Students will make math drawings and connect them to the steps in the new methods in order to explain these new methods using the place-value language supported by the drawings. First teachers have students quickly build up meanings for quantities by using a sheet of centimeter dot paper (see *How to make math drawings for hundreds, tens, and ones* on karenfusonmath.net or karenfusonmath.com on the Math Expressions Users page). On this sheet, students draw columns through ten dots to make ten-sticks. They draw around ten such ten-sticks to make hundred-boxes composed of 100 dots and ten tens. Students then make a math drawing for a 3-digit addition problem using New Groups Below and then for Show All Totals. Students discuss advantages and disadvantages of each method and compare these to the method they use. The goal is not necessarily to get them to change methods but to develop understanding of written methods related to place value. Some students do change their method after such explanations, and many students say that they are more comfortable with computation when they can explain it.

The area model is very powerful for multidigit multiplication and division. Connecting written multiplication and division algorithms to an area model is very helpful to students especially if this was not how they were taught. Teaching students the accessible multiplication and division algorithms can be very helpful for all students in understanding what they are doing with the algorithms they use. Learning and explaining these easier methods is especially helpful to students who are struggling or making errors with less accessible algorithms.

10. Extending These Accessible Algorithms to Decimals

These addition and subtraction accessible standard algorithms extend easily to decimals (Beckmann & Fuson, 2014, in *Presentations* and Numbers Base Ten, Place Value Parts 4, 5, 6, 7 in *Teaching Progressions*, both on karenfusonmath.com and karenfusonmath.net). First, students see a place-value chart as in Figure 15. They discuss all the relationships they see and especially that numbers are symmetric around the ones place, with larger products of tens moving to the left and smaller products of tenths moving to the right. Money can be used to support meanings of place values as shown on the chart.

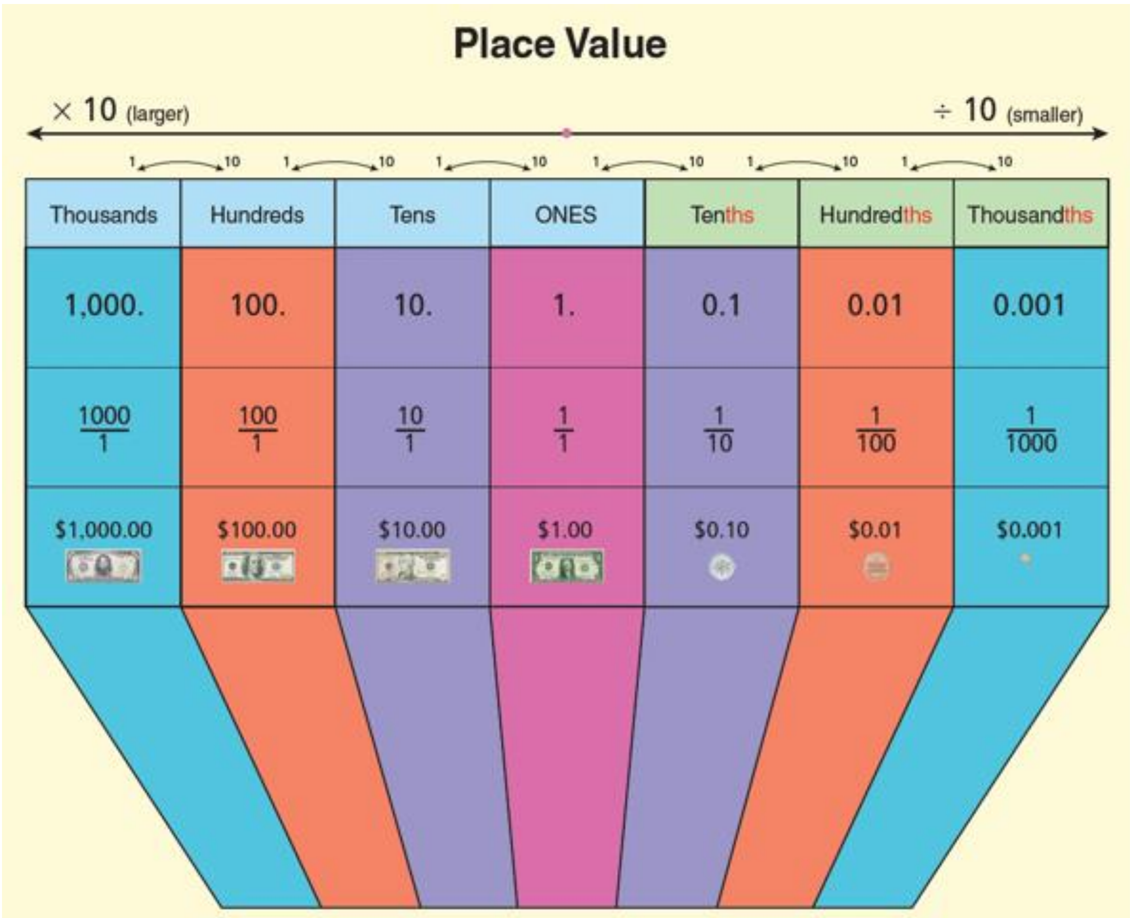


Figure 15. A place value poster showing relationships between whole numbers and decimals

The approach for addition and subtraction begins by discussing how one added or subtracted whole numbers: one added or subtracted like whole number place values. So, one needs to do the same thing for decimal places: add or subtract tenths to or from tenths, hundredths to or from hundredths, and thousandths to or from thousandths. The grouping or ungrouping works as it does for whole numbers because the bidirectional one-for-ten trades are symmetric around the ones place and work within the decimal place values as they did in the whole number place values.

For multiplying and dividing, students can envision one factor sitting on a place-value chart and then explore how multiplying by a whole number or by a decimal shifts that number left or right on the place-value chart. Multiplying by one decimal place (by tenths) will move that factor one place to the right, one place smaller, because students will be taking one-tenth of each number in each place in that factor. More decimal places in the multiplier can be thought of as repeated shifts of the multiplied factor to the right as each number has one-tenth of it taken as many times as there are decimal places in the multiplier. This process makes one factor smaller by moving it to the right while the multiplying factor becomes a whole number. The product of these two factors will have as many decimal places as in the shifted multiplied factor, which is the sum of the decimal places in both factors. These shifts on the place-value places are shown in Figures 16, 17, and 18.

Jordan earns $243 a week. What happens to his earnings after 10 weeks, 100 weeks, and 1,000 weeks? Why does this nice pattern occur?

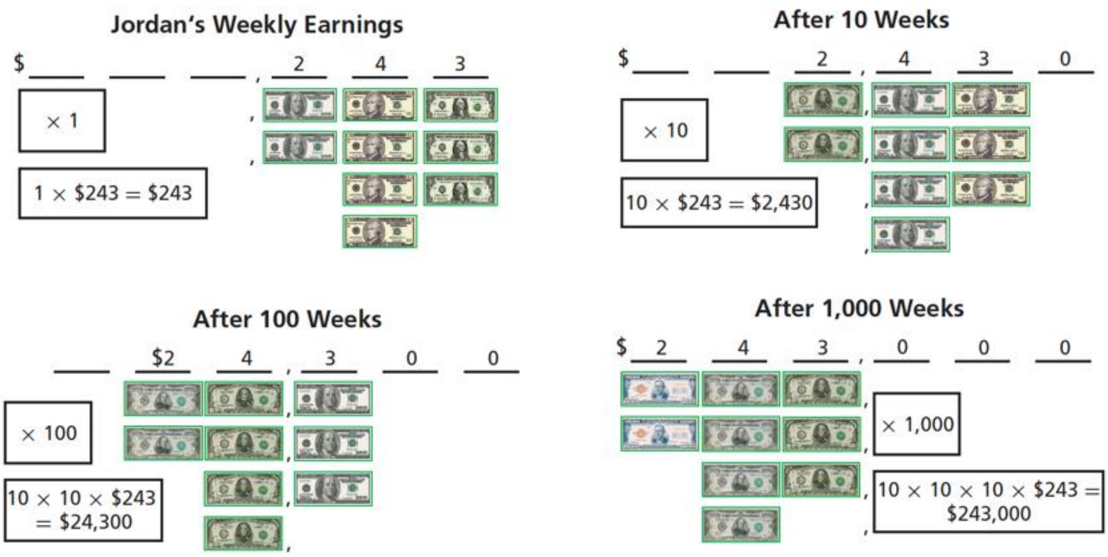


Figure 16. The whole number shifts to the left (gets bigger) when multiplying it by whole numbers over ten

It costs $0.412 (41 and 2/10 cents) for a factory to make a Red Phantom Marble. What happens to the cost after 10 weeks, 100 weeks, and 1,000 weeks? Why does this nice pattern occur?

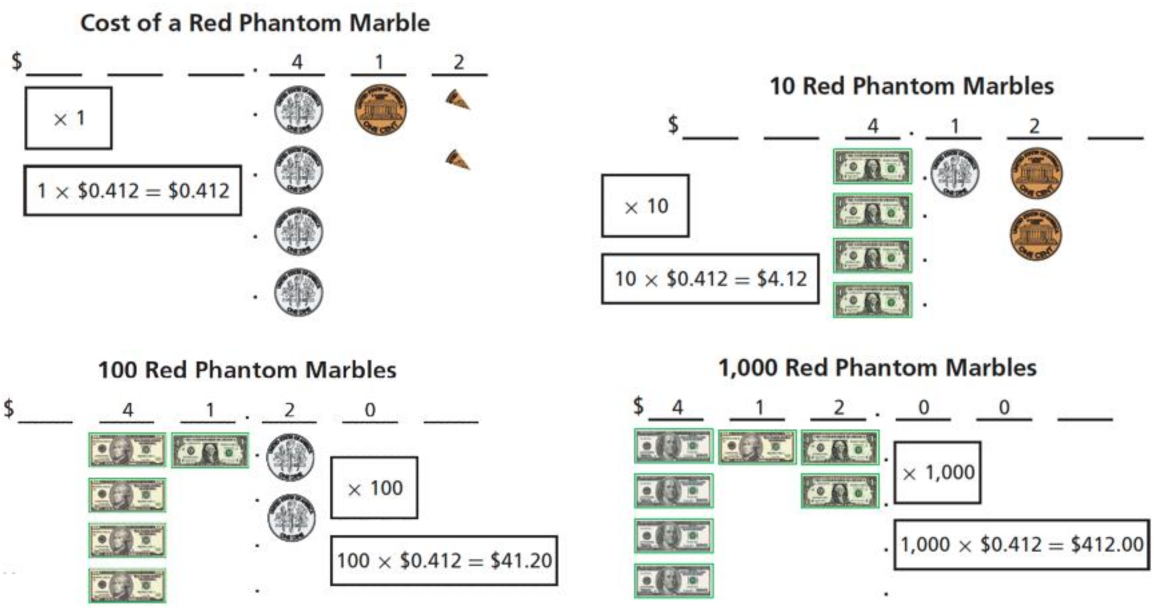


Figure 17. The decimal number shifts to the left (gets bigger) when multiplying it by whole numbers over ten

Leon earns $213 a week. He saves every month. How do his savings relate to his earnings after 10 weeks and 100 weeks. Why do his earnings shift to the right to show his savings?

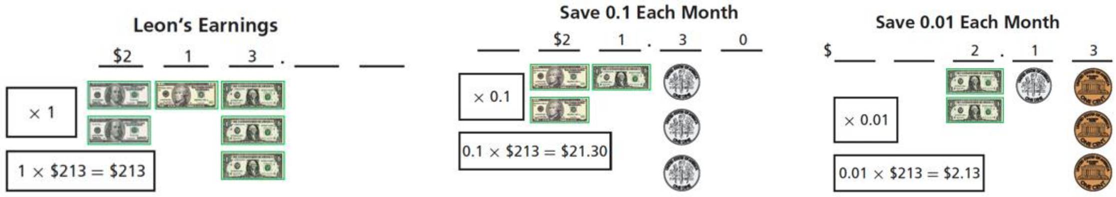


Figure 18. The whole number shifts to the right (gets smaller) when multiplying it by a decimal number because you are taking one-tenth of the numbers in each place

Division by a decimal shifts the dividend to the left (it becomes larger) because one is finding how many tenths are in each number in each place in the dividend. This shift takes place for each place in the divisor until the divisor becomes a whole number. These shifts can be marked in the long division format by moving the decimal points to the right, but we found that it was important to continue to have students discuss this process as shifting both numbers to the left, each getting bigger until the divisor is a whole number. Students also like to think of dividing in fraction form: They can see the process of writing the dividend above the divisor and multiplying that fraction by a 1 in a form (10/10 or 100/100 or 1000/1000) that will change the divisor to a whole number. These two methods together help students to understand the division process of both known numbers (the product and the known factor) getting larger the same number of places.

This approach for both multiplication and division is to change the multiplier or the divider to a whole number because we know how to multiply and divide by whole numbers. Doing several such examples allows students to reflect on the patterns involved and formulate some general rule. For multiplication we can change the multiplying number to a whole number by multiplying it by ten for each decimal position it has. So, for example, you can change 0.276 to a whole number by multiplying it by ten three times (or multiplying by 1,000) because each multiplication by 10 moves 0.276 one place to the left so it finally becomes 276. But such multiplying changes the original problem, so one needs to divide the other factor by 10 three times to keep the problem the same. This moves that factor three places to the right (three places smaller). We can see this compensatory process for 0.276 x 3.24 like this:

$$0.276 \times 3.24 = 0.276 \times 3.24 \times 1 = 0.276 \times 3.24 \times 1{,}000 \div 1{,}000$$

$$= 0.276 \times 1{,}000 \times 3.24 \div 1{,}000 = 276 \times 0.00324$$

Doing this several times with different problems can lead to the understanding that one can multiply as if both numbers were whole numbers and then count the total number of decimal places in both factors to find the number of decimal places in the product.

Discussing such strategies for changing a multiplication or division by a decimal into a multiplication or division by a whole number can help students relate and solidify understandings of relationships between whole numbers and decimals. Students also get to see the mathematical process of turning a problem you do not know how to solve into a case that you can solve (multiplying or dividing by a whole number).

11. Bringing Family Methods Into the Classroom

Students can be asked to interview family members about methods they know and then students can bring such methods into the classroom to discuss and explain. It has been our experience that such methods are rarely understood even by teachers who may have learned them (e.g., Ron, 1998) because parents and teachers usually were not taught with any understanding but only taught procedures to be memorized. But if students make math drawings to help them explain these methods and work together, they usually can figure out why a method works. Students also can look for methods on the web and in books and bring them into the classroom. Doing all of these things can change multidigit computation to being an interesting place for thinking rather than being a source of anxiety or boredom.

12. Conclusion

The algorithms taught for multidigit computations vary around the world and even within some countries. In some countries particular methods may be considered as the only methods that should be taught. Because the methods commonly taught in the United States were so problematic, Fuson undertook research studies to find methods that were easier to understand and that created fewer errors. Algorithms that students developed were gathered and tried in many other classrooms. Mathematical criteria for identifying better methods were established through cognitive and mathematical analysis of crucial conceptual aspects. This effort identified for each kind of operation two methods that were accessible and mathematically desirable. These methods are described in this paper so that educators anywhere can consider whether they might be better than the methods presently taught. The emphasis throughout is using math drawings made by students so that the steps in the algorithms take on place-value meanings and can be explained using place-value language for each drawn step.

Acknowledgements

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